Roots of Equations - Fixed Point Method

M311 - Chapter 2

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Lesson Outline

Fixed Point Iteration

Fixed Point Iteration

If the equation, $f(x) = 0$ is rearranged in the form

$$x = g(x)$$

then an iterative method may be written as

$$x_{n+1} = g(x_n) \qquad n = 0, 1, 2, \ldots \tag{1}$$

where n is the number of iterative steps and x_0 is the initial guess. This method is called the **Fixed Point Iteration** or **Successive Substitution Method.**

Definition of Fixed Point

If $c = g(c)$, the we say c is a **fixed point** for the function $g(x)$.

Theorem

Fixed Point Theorem (FPT)

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all x in $[a, b]$. Suppose, in addition, that $g'(x)$ exists on (a, b). Assume that a constant K exists with

$$|g'(x)| \leq K < 1, \quad \textit{for all } x \textit{ in } (a, b)$$

Assume that c in (a, b) is a fixed point for g. Then if x_0 is any point in (a, b), the sequence

$$x_{n+1} = g(x_n) \qquad\qquad n = 0, 1, 2, \ldots$$

converges to the unique fixed point c.(Proof - B& F page 59)

Example: Given $f(x) = x^3 - 7x + 2 = 0$ in [0,1].
Find a sequence that $\{x_n\}$ that converges to the root of $f(x) = 0$ in [0,1].
Answer: Rewrite $f(x) = 0$ as $x = \frac{1}{7}(x^3 + 2)$. Then $g(x) = \frac{1}{7}(x^3 + 2)$ and $g'(x) = \frac{3x^2}{7} < \frac{3}{7}$ for all $x \in [0, 1]$. Hence, by the FPT the sequence $\{x_n\}$ defined by

$$x_{n+1} = \frac{1}{7}(x^3 + 2)$$

converges to a root of $x^3 - 7x + 2 = 0$

Example: Solve $f(x) = x^3 - x - 1 = 0$ on $(1, 2)$.
Answer: Note $f(-1) = -1$ and $f(2) = 5$, $\therefore$ by the IVT a root exists on (1,2). Set $g(x) = (1 + x)^{\frac{1}{3}}$. Note that $g'(x) = \frac{1}{3}(1 + x)^{-2/3}$. So, on (1,2) we have

$$\frac{1}{3(1+2)^{2/3}} < g'(x) < \frac{1}{3(1+1)^{2/3}}$$

$$\therefore 0 < g'(x) < \frac{1}{3(2^{2/3})} = K$$

and $|g'(x)| \leq K < 1$ on (1,2). By the FPT the sequence

$$x_{n+1} = (1 + x_n)^{\frac{1}{3}}$$

will converge to a fixed point on (1,2).

$$
\begin{aligned}
x_{n+1} &= (1+x_n)^{\frac{1}{3}}\ , \qquad x_0 = 1.3 \\
x_0 &= 1.3 \\
x_1 &= 1.320006122 \\
x_2 &= 1.323822354 \\
x_3 &= 1.324547818 \\
x_4 &= 1.324685639 \\
&\vdots \\
x_{11} &= 1.324717957 \\
x_{12} &= 1.324717957 \\
x_{13} &= 1.324717957
\end{aligned}
$$

Example 2 & 3 of B & F. (Page 57-58)

Definition

Suppose that $\{x_n\}$ is a sequence of numbers generated by an algorithm, and the limit of the sequence is s. If

$$\lim_{n\to\infty} \frac{|x_{n+1} - s|}{|x_n - s|^p} = K \qquad , K \neq 0$$

for some positive constants K and p, the we say that the sequence $\{x_n\}$ converges to s with p being the **order of convergence.**

If $p = 1$, convergence is linear.
If $p = 2$, convergence is quadratic.
Larger values of p imply faster rates of convergence.