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On the Rate of Decay of the Concentration Function of the Sum of Independent Random Variables

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Abstract. Let $X_1, \ldots, X_n$ be *i.i.d.* integral valued random variables and S_n their sum. In the case when X_1 has a moderately large tail of distribution, Deshouillers, Freiman and Yudin gave a uniform upper bound for $\max_{k\in\mathbb{Z}} \Pr\{S_n = k\}$ (which can be expressed in term of the Lévy Doeblin concentration of S_n), under the extra condition that X_1 is not essentially supported by an arithmetic progression. The first aim of the paper is to show that this extra condition cannot be simply ruled out. Secondly, it is shown that if X_1 has a very large tail (larger than a Cauchy-type distribution), then the extra arithmetic condition is not sufficient to guarantee a uniform upper bound for the decay of the concentration of the sum S_n. Proofs are constructive and enhance the connection between additive number theory and probability theory.

Key words: Lévy concentration, sum of i.i.d. random variables, arithmetic progression, additive number theory

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1. Introduction

In order to get a measure of the dispersion (or the concentration) of real random variables, even when it is not integrable, P. Lévy introduced the notion of concentration:

$$Q(X, \lambda) = \sup_t P\{t \le X \le t + \lambda\}.$$

It is quite natural to expect that the concentration of the sum S_n of n independent identically distributed (*i.i.d.*) random variables with the same law as X will decrease as n is increasing and tend to zero as n goes to infinity, as soon as X is not almost surely constant. This was indeed proved by P. Lévy. The monograph [4] by W. Hengartner and R. Theodorescu gives many explicit results of this type, due mainly to Doeblin, Lévy, Kolmogorov, Rogozin, Esseen, Kesten. One of the features of these delicate upper bounds, is that, on the n aspect, the rate of decay of the concentration of S_n is majorized by $n^{-1/2}$.

However, if one considers independent random variables following a Cauchy law, one readily sees that in this case, the rate of decay of the concentration of S_n has the speed n^{-1}. The only result in [4] connecting the concentration of S_n with a rate of decay quicker than $n^{-1/2}$ is a lower bound due to Esseen (cf. Theorem 2.3.2, p.74), a corollary of which is quoted below as Lemma 1.

The question of the upper bound has recently attracted the attention of a few authors. J-M. Deshouillers, G.A. Freiman and A.A. Yudin [1] revived the subject by showing how one could get such results by combining Bochner theorem on the positivity of the Fourier transform of a positive measure, additive number theory and classical Fourier techniques. One of their results is the following.

Theorem 1 (Deshouillers, Freiman, Yudin). *Let* $\frac{\log 4}{\log 3} < \sigma < 2$*; let* $\varepsilon > 0$*,* $A \geq 1$*,* $a > 0$ *and let* $X_1, \ldots, X_k, \ldots$ *be i.i.d. integral valued random variables such that*:

$$\forall L \geq A: \quad 1 - Q(X_1, L) \geq aL^{-\sigma}. \tag{1}$$

$$\max_{q \geq 2} \max_{s \bmod q} P\{X_1 \equiv s \bmod q\} \leq 1 - \varepsilon. \tag{2}$$

Then we have for $n \geq 1$

$$Q(X_1 + \cdots + X_n, 1/2) \leq cn^{-1/\sigma}, \tag{3}$$

where c is a constant that depends on σ, ε, A *and* a *at most.*

Subsequently, A.S. Fainleib [3] considered the case when the random variables are not necessarily integral valued and the right hand side of (1) is of a more general type. His Corollary 1 is read as follows.

Theorem 2 (A.S. Fainleib). *Let H be an increasing continous fuction on* $[0, \infty)$ *such that* $H(0) = 0$ *and*

$$H(4u) \leq 3H(u) \tag{4}$$

for u sufficiently small. Let G be its inverse function and let $X_1, \ldots, X_k, \ldots$ *be i.i.d. real valued random variables such that*

$$\forall L \geq A: \quad 1 - Q(X_1, L) \geq G(L^{-1}). \tag{5}$$

Then we have for $n \geq 1$

$$Q(X_1 + \cdots + X_n, 1/2) \leq cH(n^{-1}), \tag{6}$$

where c is a constant that depends on H, A *and the law of* X_1 *at most.*

The comment that follows the statement of Corollary 1 in [3], which is, up to a necessary change in notation "Deshouillers et al. considered the last statement for the particular case in which X_i are integral valued and satisfy a certain arithmetical condition and $H(x) = bx^{\beta/2}$,

$b > 0$, $1 < \beta < \log_2 3$", seems to miss one point that was addressed to in Theorem 1: on which property of X_1 does the constant in (6) depend ? The answer given by the authors of [1] was that, in the case when X_1 has a moderately large tail of distribution (Condition (1) or (5)), a sufficient condition is that X_1 should not be concentrated in arithmetic progressions. Although condition (2) is obviously not necessary *stricto sensu*, the constructions we perform in this paper strengthen our belief that some arithmetic property in the style of (2) is necessary for a uniform upper bound as (3) to hold.

Our first aim in this paper is to show that, in the uniform phrasing of Theorem 1, the condition (2) cannot be simply ruled out.

Theorem 3 . *Let* $0 < \sigma < 2$*. There exists a family of i.i.d. real valued random variables* $X_1^{(n)}, \ldots, X_n^{(n)}$, $(n = 1, 2, \ldots)$, *satisfying*

$$\forall L \geq 2: \quad 1 - Q\big(X_1^{(n)}, L\big) \geq \frac{1}{10} L^{-\sigma}, \tag{7}$$

and

$$n^{1/\sigma} Q\big(X_1^{(n)} + \cdots + X_n^{(n)}, 1/2\big) \tag{8}$$

tends to infinity with n.

Although we shall construct the random variables $X_k^{(n)}$ with integral values, it is clear that a little smoothing permits to show that one may request that the the random variables satisfying Theorem 3 are continuous and their support is actually $\mathbb{R}$.

The authors of [1] suggest that further progress in the inverse additive theory on the torus, in the spirit of the work [5] of V. Lev would permit to extend the range of σ in Theorem 1 to the interval [1, 2]. Our second aim is to show that 1 is a natural bound for Theorem 1, by showing the following

Theorem 4. *Let* $0 < \sigma < 1$*. There exists a family of i.i.d. integral valued random variables* $X_1^{(n)}, \ldots, X_n^{(n)}$, $(n = 1, 2, \ldots)$, *satisfying*

$$\forall L \geq 2: \quad 1 - Q\big(X_1^{(n)}, L\big) \geq \frac{1}{10} L^{-\sigma}, \tag{9}$$

$$\max_{q \geq 2} \max_{s \bmod q} P\{X_1 \equiv s \bmod q\} = \frac{1}{2} \tag{10}$$

and

$$n^{1/\sigma} Q\big(X_1^{(n)} + \cdots + X_n^{(n)}, 1/2\big) \tag{11}$$

tends to infinity with n.

2. Proof of Theorem 3

2.1.

The starting point of our construction is a probability measure $\mu^{(1)}$ with a tail $\mu^{(1)}(\mathbb{R}\backslash]$-L,L[) of the order $L^{-\sigma}$. For $0 < \sigma < 2$, we define the probability measure $\mu^{(1)}$ by the relations

$$\mu^{(1)}(0) = \mu^{(1)}(-1) = \mu^{(1)}(1) = 1/5 \tag{12}$$

and, for $k \geq 2$

$$\mu^{(1)}(k) = \mu^{(1)}(-k) = w(k, \sigma) = \frac{1}{5}\int_{k-1}^{k} \sigma x^{-1-\sigma} dx = \frac{1}{5}\left(\frac{1}{(k-1)^{\sigma}} - \frac{1}{k^{\sigma}}\right). \tag{13}$$

It is easily seen that $\mu^{(1)}$ is a probability measure and satisfies

$$\forall L \geq 1 : \mu^{(1)}([-L, L]) \leq 1 - \frac{2}{5}L^{-\sigma} \tag{14}$$

We shall now construct the probability measure $\mu^{(n)}$ by pushing some masses of $\mu^{(1)}$ apart from 0, so that we only increase the size of its tail, and (14) will still be satisfied by $\mu^{(n)}$. In order to benefit from this construction also in the proof of Theorem 4, we introduce further parameters δ, r and τ (which can be explicitly chosen in term of σ) such that

$$0 < \sigma < r < \delta \leq 2, \tag{15}$$

τ is a positive integer and

$$\sigma\tau \geq 2 \quad \text{and} \quad \delta\sigma\tau/(\delta - \sigma) > 1 + 1/r. \tag{16}$$

As the Referee has pointed out, one may take the value $\delta = 2$ for the proof of Theorems 3 and 4. For a forthcoming use of this construction, we keep the possibility for δ to be chosen as close to σ as possible.

We further let $K_1 = 1$ and define K_n for $n \geq 2$ by

$$K_n = \lfloor n^{\tau\sigma/(\delta-\sigma)} \rfloor. \tag{17}$$

For $n \geq 2$ we define the probability measure $\mu^{(n)}$ in the following way

$$\mu^{(n)}(0) = \mu^{(n)}(-n^{\tau}) = \mu^{(n)}(n^{\tau}) = \frac{1}{5}; \tag{18}$$

for $2 \leq k \leq K_n$, we let

$$\mu^{(n)}(kn^{\tau}) = \mu^{(n)}(-kn^{\tau}) = w(k, \delta) = \frac{1}{5}\int_{k-1}^{k} \delta x^{-1-\delta} dx; \tag{19}$$

$$\mu^{(n)}(n^\tau(1+K_n)) = \mu^{(n)}(-n^\tau(1+K_n)) = \frac{1}{5}\left(\frac{1}{K_n^\delta} - \frac{1}{(n^\tau(1+K_n))^\sigma}\right); \tag{20}$$

for $l > n^\tau(1+K_n)$, we let

$$\mu^{(n)}(-l) = \mu^{(n)}(l) = w(l,\sigma) = \frac{1}{5}\int_{l-1}^{l} \sigma x^{-1-\sigma}dx, \tag{21}$$

and for those l's in $[-n^\tau(1+K_n), n^\tau(1+K_n)]$ which are not multiple of n^τ, we let $\mu^{(n)}(l) = 0$.

Since, by the definition (17) of K_n we have $K_n \geq 1$, the measure $\mu^{(n)}$ is well defined. Let us verify that it is a probability measure. We first show that it is a positive measure, which amounts to checking that the RHS of (20) is non-negative: we have indeed

$$K_n \leq n^{\tau\sigma/(\delta-\sigma)},$$

whence

$$K_n{}^\delta \leq n^{\tau\sigma\delta/(\delta-\sigma)} \leq n^{\tau\sigma}.n^{\tau\sigma^2/(\delta-\sigma)} < (n^\tau(1+K_n))^\sigma.$$

The second point to check is that the total mass is 1; we have

$$\mu^{(n)}(\mathbb{R}) = \frac{3}{5} + \frac{2}{5}\int_1^{K_n} \delta x^{-1-\delta}dx + \frac{2}{5}\left(\frac{1}{K_n^\delta} - \frac{1}{(n^\tau(1+K_n))^\sigma}\right) + \frac{2}{5}\int_{n^\tau(1+K_n)}^{\infty} \sigma x^{-1-\sigma}dx,$$

which is readily seen to be equal to 1.

2.2.

For $n = 1, 2, \ldots$, let $X_1^{(n)}, \ldots, X_n^{(n)}$, be independent random variables, the common law of which is $\mu^{(n)}$. We prove here that they satisfy (7). Let $L \geq 1$ and I be a closed interval with length L.

If I does not contain 0, or if $L \leq 2$, we have

$$P\left\{X_1^{(n)} \in I\right\} \leq \frac{4}{5} \leq 1 - \frac{1}{5}L^{-\sigma}.$$

If I contains 0 and $L > 2$, we let $I = [a, b]$ and we can assume that $0 \leq |a| \leq b \leq L$, since the distribution of $X_1^{(n)}$ is even.

We shall use $P\{X_1^{(n)} \in I\} \leq P\{X_1^{(n)} \in [-b, b]\}$ and consider 3 cases according to the size of b.

1. If $b \geq n^\tau(1+K_n)$, we have

$$P\{X_1^{(n)} \notin [-b,b]\} = 1 - \mu^{(1)}([-b,b])$$

since $\mu^{(1)}$ and $\mu^{(n)}$ respectively give the same masses to points which are outside $[-n^\tau(1+K_n), n^\tau(1+K_n)]$. We thus have

$$\begin{aligned} P\{X_1^{(n)} \in [-b,b]\} &= P\{X_1^{(1)} \in [-b,b]\} \\ &= \frac{3}{5} + 2\sum_{2\leq k\leq [b]} \mu^{(1)}(k) \\ &\leq \frac{3}{5} + \frac{2}{5}\int_1^{[b]} \sigma x^{-1-\sigma}\,dx \\ &\leq 1 - \frac{1}{10}L^{-\sigma}. \end{aligned}$$

2. If $b < 2n^\tau$, we have

$$P\{X_1^{(n)} \in [-b,b]\} \leq \frac{3}{5} \leq 1 - \frac{1}{10}L^{-\sigma}.$$

3. If $2n^\tau \leq b < n^\tau(1+K_n)$, we have

$$\begin{aligned} P\{X_1^{(n)} \in [-b,b]\} &\leq \frac{3}{5} + 2\sum_{2\leq k\leq [b/n^\tau]} w(k,\delta) \\ &= 1 - \frac{2}{5}\left(\left[\frac{b}{n^\tau}\right]\right)^{-\delta}. \end{aligned}$$

Since $b/n^\tau < 1+K_n \leq 2n^{\tau\sigma/(\delta-\sigma)}$, we have

$$(b/n^\tau)^{(\delta-\sigma)} \leq 2^{(\delta-\sigma)} n^{\tau\sigma} \leq 4n^{\tau\sigma},$$

whence

$$(b/n^\tau)^\delta \leq 4n^{\tau\sigma}(b/n^\tau)^\sigma \leq 4b^\sigma \leq 4L^\sigma$$

from which we get

$$P\{X_1^{(n)} \in [-b,b]\} \leq 1 - \frac{1}{10}L^{-\sigma}.$$

This proves (7).

2.3.

We now prove Relation (8). In order to show that the concentration of the sum of the $X_k^{(n)}$ is large, we'll make use of the following result, which is a direct corollary of a result of C.G. Esseen (cf. [2] or [4])

Lemma 1. *Let* $0 < r < 2$ *and let* $Y_1, \dots, Y_n$ *be i.i.d. random variables. We have*

$$Q(Y_1 + \cdots + Y_n, 1/2) \geq \frac{r}{48} \min(2, (n\mathbb{E}(|Y_1|^r))^{-1/r}). \tag{22}$$

Let φ_n be the normalized characteristic function of $X_1^{(n)}$, *i.e.* $\mathbb{E}(\exp(2\pi i t X_1^{(n)}))$. Letting $p_h = P\{X_1^{(n)} = h\}$ and $e(u) = \exp(2\pi i u)$, we can write

$$\varphi_n(t) = \sum_{|l| \leq K_n} p_{ln^\tau} e(ln^\tau t) + \sum_{|l| > n^\tau K_n} p_l e(lt) = \psi_n(t) + R_n(t). \tag{23}$$

When n is large enough, we have

$$|R_n(t)| \leq \frac{2}{5} K_n^{-\delta} = \frac{2}{5}\left[n^{\tau\sigma/(\delta-\sigma)}\right]^{-\delta} \leq n^{-\delta\tau\sigma/(\delta-\sigma)};$$

since $\max(|\psi_n(t)|, |\varphi_n(t)|) \leq 1$, we have

$$\left|\varphi_n^n(t) - \psi_n^n(t)\right| \leq n|R_n(t)| \leq n^{1-\delta\tau\sigma/(\delta-\sigma)}. \tag{24}$$

Let us define

$$q_l = \begin{cases} \dfrac{1}{1 - R_n(0)} p_{ln^\tau} & \text{for } |l| \leq K_n \\ 0 & \text{for } |l| > K_n \end{cases}.$$

Since $\sum_{|l| \leq K_n} q_l = 1$, we can define a family of i.i.d. random variable $Z_1^{(n)}, \dots, Z_n^{(n)}$ such that $P\{Z_1^{(n)} = l\} = q_l$ for any l in $\mathbb{Z}$. Their normalized characteristic functions $\tilde{\psi_n}$ satisfy

$$\tilde{\psi_n}(t) = \sum_{|l| \leq K_n} q_l e(lt) = \frac{1}{1 - R_n(0)} \psi_n\left(\frac{t}{n^\tau}\right).$$

By (24) and a similar inequality connecting $\tilde{\psi_n}^n$ and ψ_n^n, we have

$$\left|\varphi_n^n(t) - \tilde{\psi_n}^n(n^\tau t)\right| \leq 2n^{1-\delta\tau\sigma/(\delta-\sigma)}.$$

By integrating over [0, 1], we get

$$\left| \int_0^1 \varphi_n^n(t)\, dt - \int_0^1 \tilde{\psi_n}^n(n^\tau t)\, dt \right| \leq 2n^{1-\delta\tau\sigma/(\delta-\sigma)};$$

we change the variable in the second integral and use the 1-periodicity of $\tilde{\psi_n}$; we thus obtain

$$\left| \int_0^1 \varphi_n^n(t)\, dt - \int_0^1 \tilde{\psi_n}^n(t)\, dt \right| \leq 2n^{1-\delta\tau\sigma/(\delta-\sigma)},$$

which is also

$$\left|P\{X_1^{(n)}+\cdots+X_n^{(n)}=0\}-P\{Z_1^{(n)}+\cdots+Z_n^{(n)}=0\}\right|\le 2n^{1-\delta\tau\sigma/(\delta-\sigma)}, \tag{25}$$

By the remark at the top of p.10 of [4], (25) is equivalent to

$$\left|Q\big(X_1^{(n)}+\cdots+X_n^{(n)},1/2\big)-Q\big(Z_1^{(n)}+\cdots+Z_n^{(n)},1/2\big)\right|\le 2n^{1-\delta\tau\sigma/(\delta-\sigma)}. \tag{26}$$

We recall that $r\in]\sigma,\delta[$; thus

$$\mathbb{E}\big(|Z^{(n)}|^r\big)=(1-R_n(0))^{-1}\left(\frac{2}{5}+\frac{2}{5}\sum_{2\le l\le K_n}l^r\left(\frac{1}{(l-1)^\delta}-\frac{1}{l^\delta}\right)\right),$$

tends to a constant C_1 as n tends to infinity. By Esseen's Lemma 1, there exists a constant C_2 such that for all n we have

$$Q\big(Z_1^{(n)}+\cdots+Z_n^{(n)},1/2\big)\ge C_2n^{-1/r}. \tag{27}$$

Relation (8) then comes from (27) and (26), thanks to (16). Theorem 3 is thus proved.

3. Proof of Theorem 4

In this section, we assume $0<\sigma<1$ and let the parameters δ, r and τ satisfy Condition (16) as well as the condition

$$0<\sigma<\frac{2\sigma}{2-\sigma}<r<\delta\le 2, \tag{28}$$

which is stronger than (15).

Let the i.i.d. random variables $Y_1^{(n)},\dots,Y_n^{(n)}$ satisfy

$$P\{Y_1^{(n)}=k\}=\frac{1}{2}P\{X_1^{(n)}=k\}+\frac{1}{2}P\{X_1^{(n)}=k-1\}. \tag{29}$$

Let $L\ge 2$ be given and let I be a closed interval of length L such that $Q(Y_1^{(n)},L)=P\{Y_1^{(n)}\in I\}$. We have

$$\begin{aligned}Q\big(Y_1^{(n)},L\big)&=\frac{1}{2}P\{X_1^{(n)}\in I\}+\frac{1}{2}P\{X_1^{(n)}\in 1+I\}\\&\le\frac{1}{2}Q\big(X_1^{(n)},L\big)+\frac{1}{2}Q\big(X_1^{(n)},L\big)\le Q\big(X_1^{(n)},L\big),\end{aligned} \tag{30}$$

so that (9) comes from (1).

Let $q\ge 2$; for any s we have

$$P\{Y_1^{(n)}\equiv s \bmod q\}=\frac{1}{2}P\{X_1^{(n)}\equiv s \bmod q\}+\frac{1}{2}P\{X_1^{(n)}\equiv s-1 \bmod q\} \tag{31}$$

since the sets $P\{X_1^{(n)} \equiv s \bmod q\}$ and $P\{X_1^{(n)} \equiv s-1 \bmod q\}$ are disjoint, we have

$$P\big\{Y_1^{(n)} \equiv s \bmod q\big\} \le \frac{1}{2},$$

and equality is clearly obtained when $q = 2$; whence (10).

We finally turn our attention to the concentration of $Y_1^{(n)} + \cdots + Y_n^{(n)}$. For any k, we have

$$P\big\{Y_1^{(n)} + \cdots + Y_n^{(n)} = k\big\} = \frac{1}{2^n} \sum_{(\epsilon_1,\ldots,\epsilon_n)\in\{0,1\}^n} P\left\{X_1^{(n)} + \cdots + X_n^{(n)} = k - \sum_{i=1}^{n} \epsilon_i\right\}. \quad (32)$$

One way to prove this relation is to introduce a family of i.i.d. random variables $U_j^{(n)}$ with values in $\mathbb{Z} \times \{0, 1\}$ such that

$$P\big\{U_1^{(n)} = (\ell, i)\big\} = \frac{1}{2} P\big\{X_1^{(n)} = \ell\big\} \quad \text{for } i = 0, 1,$$

and consider $Y_j^{(n)}$ to be the sum of the components of $U_j^{(n)} (j = 1, \ldots, n)$.

Let us consider (32) with $k = [n/2]$; we get

$$\begin{aligned} P\big\{Y_1^{(n)} + \cdots + Y_n^{(n)} = [n/2]\big\} &\ge \frac{1}{2^n} \sum_{\substack{(\epsilon_1,\ldots,\epsilon_n)\in\{0,1\}^n \\ \epsilon_1+\cdots+\epsilon_n = [n/2]}} P\big\{X_1^{(n)} + \cdots + X_n^{(n)} = 0\big\} \\ &\ge \binom{n}{[n/2]} \frac{1}{2^n} P\big\{X_1^{(n)} + \cdots + X_n^{(n)} = 0\big\}. \end{aligned}$$

It is clear from 2.3 that there exists a constant K_3 such that

$$P\big\{X_1^{(n)} + \cdots + X_n^{(n)} = 0\big\} \ge K_3 n^{-1/r} \quad \text{for } n \ge 1$$

and Stirling's formula (even some weak form of it) then implies

$$Q\left(Y_1^{(n)} + \cdots + Y_n^{(n)}, \frac{1}{2}\right) \ge K_4 n^{-1/r-1/2};$$

since we chose $r > 2\sigma/(2-\sigma)$, the exponent $-1/r - 1/2$ is larger than $-1/\sigma$, whence we deduce the fact that $n^{1/\sigma} Q(Y_1^{(n)} + \cdots + Y_n^{(n)}, \frac{1}{2})$ tends to infinity with n.

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